

BIORTHOGONAL FILTERS

Patrick J. Van Fleet

Center for Applied Mathematics
University of St. Thomas
St. Paul, MN USA

PREP - Wavelet Workshop, 2007



UNIVERSITY of ST. THOMAS

OUTLINE

TODAY'S SCHEDULE

CHARACTERIZING ORTHOGONALITY

$H(\omega)$ for the Daubechies Filter

The Haar Transform - One Last Time!

Theorem

Observations

Problem in Applications

BIORTHOGONAL FILTERS

The Setup

The Solution

BIORTHOGONAL SPLINE FILTERS



TODAY'S SCHEDULE

9:00-10:15 **Lecture Six:** Daubechies Filters

10:15-10:30 **Coffee Break** (OSS 235)

10:30-11:45 **Lecture Seven:** Denoising

12:00-1:00 **Lunch** (Cafeteria)

1:15-2:30 \Rightarrow **Lecture Eight:** Biorthogonal Filters

2:30-2:45 **Coffee Break** (OSS 235)

2:45-4:15 **Computer Session Four:** Symmetry in Biorthogonal Wavelet Transform

5:30-6:30 **Dinner** (Cafeteria)



The Daubechies filter \mathbf{h} is

$$h_0 = \frac{1}{4\sqrt{2}} (1 + \sqrt{3}) \quad h_1 = \frac{1}{4\sqrt{2}} (3 + \sqrt{3})$$

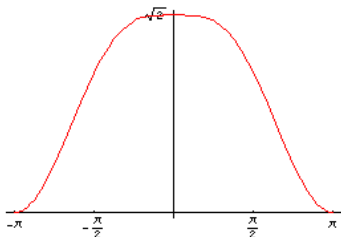
$$h_2 = \frac{1}{4\sqrt{2}} (3 - \sqrt{3}) \quad h_3 = \frac{1}{4\sqrt{2}} (1 - \sqrt{3})$$

with associated Fourier series

$$H(\omega) = h_0 + h_1 e^{j\omega} + h_2 e^{2j\omega} + h_3 e^{3j\omega}$$



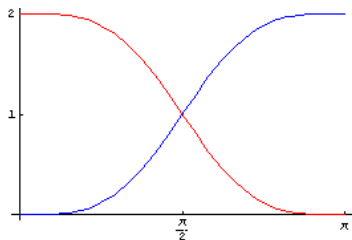
If we plot $|H(\omega)|$ over the interval $[-\pi, \pi]$, we have



$H(\omega)$ is 2π -periodic.



Now we plot $|H(\omega)|^2$ and $|H(\omega + \pi)|^2$ over the interval $[0, \pi]$:



It appears if we add these functions together, that the sum is a constant 2.



- ▶ We can verify this for the Haar filter $\mathbf{h} = (h_0, h_1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
- ▶ For $H(\omega) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e^{i\omega}$, we have

$$|H(\omega)|^2 = 2 \cos^2(\omega/2)$$

- ▶ In this case

$$\begin{aligned} |H(\omega + \pi)|^2 &= 2 \cos^2(\omega/2 + \pi/2) \\ &= 2 \sin^2(\omega/2) \end{aligned}$$

- ▶ Thus we see that

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$



- ▶ We can verify this for the Haar filter $\mathbf{h} = (h_0, h_1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
- ▶ For $H(\omega) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e^{i\omega}$, we have

$$|H(\omega)|^2 = 2 \cos^2(\omega/2)$$

- ▶ In this case

$$\begin{aligned} |H(\omega + \pi)|^2 &= 2 \cos^2(\omega/2 + \pi/2) \\ &= 2 \sin^2(\omega/2) \end{aligned}$$

- ▶ Thus we see that

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$



- ▶ We can verify this for the Haar filter $\mathbf{h} = (h_0, h_1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
- ▶ For $H(\omega) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e^{i\omega}$, we have

$$|H(\omega)|^2 = 2 \cos^2(\omega/2)$$

- ▶ In this case

$$\begin{aligned} |H(\omega + \pi)|^2 &= 2 \cos^2(\omega/2 + \pi/2) \\ &= 2 \sin^2(\omega/2) \end{aligned}$$

- ▶ Thus we see that

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$



- ▶ We can verify this for the Haar filter $\mathbf{h} = (h_0, h_1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.
- ▶ For $H(\omega) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} e^{i\omega}$, we have

$$|H(\omega)|^2 = 2 \cos^2(\omega/2)$$

- ▶ In this case

$$\begin{aligned} |H(\omega + \pi)|^2 &= 2 \cos^2(\omega/2 + \pi/2) \\ &= 2 \sin^2(\omega/2) \end{aligned}$$

- ▶ Thus we see that

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 2$$



THEOREM

Suppose $H(\omega) = \sum_k h_k e^{ik\omega}$. Then

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = H(\omega)\overline{H(\omega)} + H(\omega + \pi)\overline{H(\omega + \pi)} = 2 \quad (1)$$

if and only if

$$\sum_k h_k^2 = 1$$

and for $m \neq 0$,

$$\sum_k h_k h_{k-2m} = 0$$

In other words, if $H(\omega)$ satisfies (1), it gives rise to an orthogonal wavelet transform matrix.



- ▶ The theorem gives us a way to completely characterize orthogonal wavelet transforms.
- ▶ The Fourier series $G(\omega)$ of the highpass filter \mathbf{g} can be obtained from $H(\omega)$:

$$G(\omega) = e^{iL\omega} \overline{H(\omega + \pi)}$$

where L is any odd integer.

- ▶ Simplifying the above expression allows us to extract the highpass filter coefficients:

$$g_k = (-1)^k h_{L-k}$$

- ▶ Conditions other than (in addition to) $H^{(n)}(\pi) = 0$ can be added to the orthogonality conditions to build wavelet filters suited to specific tasks.



- ▶ The theorem gives us a way to completely characterize orthogonal wavelet transforms.
- ▶ The Fourier series $G(\omega)$ of the highpass filter \mathbf{g} can be obtained from $H(\omega)$:

$$G(\omega) = e^{iL\omega} \overline{H(\omega + \pi)}$$

where L is any odd integer.

- ▶ Simplifying the above expression allows us to extract the highpass filter coefficients:

$$g_k = (-1)^k h_{L-k}$$

- ▶ Conditions other than (in addition to) $H^{(n)}(\pi) = 0$ can be added to the orthogonality conditions to build wavelet filters suited to specific tasks.



- ▶ The theorem gives us a way to completely characterize orthogonal wavelet transforms.
- ▶ The Fourier series $G(\omega)$ of the highpass filter \mathbf{g} can be obtained from $H(\omega)$:

$$G(\omega) = e^{iL\omega} \overline{H(\omega + \pi)}$$

where L is any odd integer.

- ▶ Simplifying the above expression allows us to extract the highpass filter coefficients:

$$g_k = (-1)^k h_{L-k}$$

- ▶ Conditions other than (in addition to) $H^{(n)}(\pi) = 0$ can be added to the orthogonality conditions to build wavelet filters suited to specific tasks.



- ▶ The theorem gives us a way to completely characterize orthogonal wavelet transforms.
- ▶ The Fourier series $G(\omega)$ of the highpass filter \mathbf{g} can be obtained from $H(\omega)$:

$$G(\omega) = e^{iL\omega} \overline{H(\omega + \pi)}$$

where L is any odd integer.

- ▶ Simplifying the above expression allows us to extract the highpass filter coefficients:

$$g_k = (-1)^k h_{L-k}$$

- ▶ Conditions other than (in addition to) $H^{(n)}(\pi) = 0$ can be added to the orthogonality conditions to build wavelet filters suited to specific tasks.



- ▶ Unfortunately, orthogonal filters aren't always desirable in applications.
- ▶ Orthogonal filters can be constructed as orthogonal, finite-length, and "suitably smooth".
- ▶ For many applications, an important property possessed by filters is symmetry.
- ▶ For odd-length filters, we'll say a filter is symmetric if $h_k = h_{-k}$. Even-length filters are symmetric if $h_k = h_{1-k}$.
- ▶ Daubechies proved that the only filter is orthogonal, finite-length, suitably smooth, and symmetric is the Haar filter.
- ▶ The Haar filter suffers from the "de-coupling of data" problem.
- ▶ How to proceed?



- ▶ Unfortunately, orthogonal filters aren't always desirable in applications.
- ▶ Orthogonal filters can be constructed as orthogonal, finite-length, and “suitably smooth”.
- ▶ For many applications, an important property possessed by filters is symmetry.
- ▶ For odd-length filters, we'll say a filter is symmetric if $h_k = h_{-k}$. Even-length filters are symmetric if $h_k = h_{1-k}$.
- ▶ Daubechies proved that the only filter is orthogonal, finite-length, suitably smooth, and symmetric is the Haar filter.
- ▶ The Haar filter suffers from the “de-coupling of data” problem.
- ▶ How to proceed?



- ▶ Unfortunately, orthogonal filters aren't always desirable in applications.
- ▶ Orthogonal filters can be constructed as orthogonal, finite-length, and “suitably smooth”.
- ▶ For many applications, an important property possessed by filters is symmetry.
- ▶ For odd-length filters, we'll say a filter is symmetric if $h_k = h_{-k}$. Even-length filters are symmetric if $h_k = h_{1-k}$.
- ▶ Daubechies proved that the only filter is orthogonal, finite-length, suitably smooth, and symmetric is the Haar filter.
- ▶ The Haar filter suffers from the “de-coupling of data” problem.
- ▶ How to proceed?



- ▶ Unfortunately, orthogonal filters aren't always desirable in applications.
- ▶ Orthogonal filters can be constructed as orthogonal, finite-length, and “suitably smooth”.
- ▶ For many applications, an important property possessed by filters is symmetry.
- ▶ For odd-length filters, we'll say a filter is symmetric if $h_k = h_{-k}$. Even-length filters are symmetric if $h_k = h_{1-k}$.
- ▶ Daubechies proved that the only filter is orthogonal, finite-length, suitably smooth, and symmetric is the Haar filter.
- ▶ The Haar filter suffers from the “de-coupling of data” problem.
- ▶ How to proceed?



- ▶ Unfortunately, orthogonal filters aren't always desirable in applications.
- ▶ Orthogonal filters can be constructed as orthogonal, finite-length, and “suitably smooth”.
- ▶ For many applications, an important property possessed by filters is symmetry.
- ▶ For odd-length filters, we'll say a filter is symmetric if $h_k = h_{-k}$. Even-length filters are symmetric if $h_k = h_{1-k}$.
- ▶ Daubechies proved that the only filter is orthogonal, finite-length, suitably smooth, and symmetric is the Haar filter.
- ▶ The Haar filter suffers from the “de-coupling of data” problem.
- ▶ How to proceed?



- ▶ Unfortunately, orthogonal filters aren't always desirable in applications.
- ▶ Orthogonal filters can be constructed as orthogonal, finite-length, and “suitably smooth”.
- ▶ For many applications, an important property possessed by filters is symmetry.
- ▶ For odd-length filters, we'll say a filter is symmetric if $h_k = h_{-k}$. Even-length filters are symmetric if $h_k = h_{1-k}$.
- ▶ Daubechies proved that the only filter is orthogonal, finite-length, suitably smooth, and symmetric is the Haar filter.
- ▶ The Haar filter suffers from the “de-coupling of data” problem.
- ▶ How to proceed?



- ▶ Unfortunately, orthogonal filters aren't always desirable in applications.
- ▶ Orthogonal filters can be constructed as orthogonal, finite-length, and “suitably smooth”.
- ▶ For many applications, an important property possessed by filters is symmetry.
- ▶ For odd-length filters, we'll say a filter is symmetric if $h_k = h_{-k}$. Even-length filters are symmetric if $h_k = h_{1-k}$.
- ▶ Daubechies proved that the only filter is orthogonal, finite-length, suitably smooth, and symmetric is the Haar filter.
- ▶ The Haar filter suffers from the “de-coupling of data” problem.
- ▶ How to proceed?



- ▶ **Finite-length filters is important for speed of computation.**
- ▶ The ability to reproduce “suitably smooth” data is also important.
- ▶ Why is orthogonality important?
- ▶ It gives us a nice formula for the inverse, but once we have the inverse, we never need to re-compute it.
- ▶ Daubechies idea was to drop orthogonality.



- ▶ Finite-length filters is important for speed of computation.
- ▶ The ability to reproduce “suitably smooth” data is also important.
- ▶ Why is orthogonality important?
- ▶ It gives us a nice formula for the inverse, but once we have the inverse, we never need to re-compute it.
- ▶ Daubechies idea was to drop orthogonality.



- ▶ Finite-length filters is important for speed of computation.
- ▶ The ability to reproduce “suitably smooth” data is also important.
- ▶ Why is orthogonality important?
 - ▶ It gives us a nice formula for the inverse, but once we have the inverse, we never need to re-compute it.
 - ▶ Daubechies idea was to drop orthogonality.



- ▶ Finite-length filters is important for speed of computation.
- ▶ The ability to reproduce “suitably smooth” data is also important.
- ▶ Why is orthogonality important?
- ▶ It gives us a nice formula for the inverse, but once we have the inverse, we never need to re-compute it.
- ▶ Daubechies idea was to drop orthogonality.



- ▶ Finite-length filters is important for speed of computation.
- ▶ The ability to reproduce “suitably smooth” data is also important.
- ▶ Why is orthogonality important?
- ▶ It gives us a nice formula for the inverse, but once we have the inverse, we never need to re-compute it.
- ▶ Daubechies idea was to drop orthogonality.



- ▶ Use **two** lowpass filters instead of one!
- ▶ This is equivalent to building two wavelet transform matrices \tilde{W} and W so that $\tilde{W}^{-1} = W^T$. (Remember, if W is orthogonal, then $W^{-1} = W^T$.)
- ▶ Let \mathbf{h} and $\tilde{\mathbf{h}}$ denote the two lowpass filters and suppose $H(\omega)$, $\tilde{H}(\omega)$ are the associated Fourier series.



- ▶ Use **two** lowpass filters instead of one!
- ▶ This is equivalent to building two wavelet transform matrices \tilde{W} and W so that $\tilde{W}^{-1} = W^T$. (Remember, if W is orthogonal, then $W^{-1} = W^T$.)
- ▶ Let \mathbf{h} and $\tilde{\mathbf{h}}$ denote the two lowpass filters and suppose $H(\omega)$, $\tilde{H}(\omega)$ are the associated Fourier series.



- ▶ Use **two** lowpass filters instead of one!
- ▶ This is equivalent to building two wavelet transform matrices \tilde{W} and W so that $\tilde{W}^{-1} = W^T$. (Remember, if W is orthogonal, then $W^{-1} = W^T$.)
- ▶ Let \mathbf{h} and $\tilde{\mathbf{h}}$ denote the two lowpass filters and suppose $H(\omega)$, $\tilde{H}(\omega)$ are the associated Fourier series.



- ▶ Generalize the orthogonality condition

$$H(\omega)\overline{H(\omega)} + H(\omega + \pi)\overline{H(\omega + \pi)} = 2$$

- ▶ to

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega + \pi)\overline{H(\omega + \pi)} = 2$$



- ▶ Generalize the orthogonality condition

$$H(\omega)\overline{H(\omega)} + H(\omega + \pi)\overline{H(\omega + \pi)} = 2$$

- ▶ to

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega + \pi)\overline{H(\omega + \pi)} = 2$$



Suppose we can find $\tilde{H}(\omega)$ and $H(\omega)$ so that

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega + \pi)\overline{H(\omega + \pi)} = 2 \quad (2)$$

Then this leads to the **matrices** H and \tilde{H} satisfying $H\tilde{H}^T = I$.



If we take

$$G(\omega) = e^{i\tilde{L}\omega} \overline{\tilde{H}(\omega + \pi)} \quad \text{and} \quad \tilde{G}(\omega) = e^{iL\omega} \overline{H(\omega + \pi)}$$

where \tilde{L} and L are odd positive integers, then we can show:



▶

$$\tilde{G}(\omega)\overline{G(\omega)} + \tilde{G}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{H}(\omega)\overline{G(\omega)} + \tilde{H}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{G}(\omega)\overline{H(\omega)} + \tilde{G}(\omega + \pi)\overline{H(\omega + \pi)} = 0$$

- ▶ The first condition tells us matrices G and \tilde{G} satisfy $G\tilde{G}^T = I$.
- ▶ The next two conditions give us $G\tilde{H}^T = H\tilde{G}^T = 0$.



▶

$$\tilde{G}(\omega)\overline{G(\omega)} + \tilde{G}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{H}(\omega)\overline{G(\omega)} + \tilde{H}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{G}(\omega)\overline{H(\omega)} + \tilde{G}(\omega + \pi)\overline{H(\omega + \pi)} = 0$$

- ▶ The first condition tells us matrices G and \tilde{G} satisfy $G\tilde{G}^T = I$.
- ▶ The next two conditions give us $G\tilde{H}^T = H\tilde{G}^T = 0$.



▶

$$\tilde{G}(\omega)\overline{G(\omega)} + \tilde{G}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{H}(\omega)\overline{G(\omega)} + \tilde{H}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{G}(\omega)\overline{H(\omega)} + \tilde{G}(\omega + \pi)\overline{H(\omega + \pi)} = 0$$

- ▶ The first condition tells us matrices G and \tilde{G} satisfy $G\tilde{G}^T = I$.
- ▶ The next two conditions give us $G\tilde{H}^T = H\tilde{G}^T = 0$.



▶

$$\tilde{G}(\omega)\overline{G(\omega)} + \tilde{G}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{H}(\omega)\overline{G(\omega)} + \tilde{H}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{G}(\omega)\overline{H(\omega)} + \tilde{G}(\omega + \pi)\overline{H(\omega + \pi)} = 0$$

▶ The first condition tells us matrices G and \tilde{G} satisfy $G\tilde{G}^T = I$.

▶ The next two conditions give us $G\tilde{H}^T = H\tilde{G}^T = 0$.



▶

$$\tilde{G}(\omega)\overline{G(\omega)} + \tilde{G}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{H}(\omega)\overline{G(\omega)} + \tilde{H}(\omega + \pi)\overline{G(\omega + \pi)} = 0$$

▶

$$\tilde{G}(\omega)\overline{H(\omega)} + \tilde{G}(\omega + \pi)\overline{H(\omega + \pi)} = 0$$

- ▶ The first condition tells us matrices G and \tilde{G} satisfy $G\tilde{G}^T = I$.
- ▶ The next two conditions give us $G\tilde{H}^T = H\tilde{G}^T = 0$.



Thus we obtain matrices H , \tilde{H} , G , and \tilde{G} that allows us to form W_N and \tilde{W}_N that satisfy

$$W_N \tilde{W}_N^T = \left[\begin{array}{c} H \\ G \end{array} \right] \cdot \left[\begin{array}{c|c} \tilde{H}^T & \tilde{G}^T \end{array} \right] = \left[\begin{array}{c|c} H\tilde{H}^T & H\tilde{G}^T \\ \hline G\tilde{H}^T & G\tilde{G}^T \end{array} \right] = I_N$$

That is $\tilde{W}_N^T = W_N^{-1}$!



- ▶ It can be shown that

$$\tilde{g}_k = (-1)^k h_{L-k} \quad \text{and} \quad g_k = (-1)^k \tilde{h}_{\tilde{L}-k}$$

where \tilde{L} and L are odd positive integers. (We typically take $\tilde{L} = L = 1$.)

- ▶ We can read $\tilde{\mathbf{h}}$, \mathbf{h} from their Fourier series and build $\tilde{\mathbf{g}}$, and \mathbf{g} from them!
- ▶ $\tilde{\mathbf{h}}$ and \mathbf{h} are called **biorthogonal filters**.
- ▶ It can also be shown that there exist **symmetric** filters $\tilde{\mathbf{h}}$ and \mathbf{h} that satisfy (2).
- ▶ How do we find $\tilde{H}(\omega)$ and $H(\omega)$ that satisfy (2)?



- ▶ It can be shown that

$$\tilde{g}_k = (-1)^k h_{L-k} \quad \text{and} \quad g_k = (-1)^k \tilde{h}_{\tilde{L}-k}$$

where \tilde{L} and L are odd positive integers. (We typically take $\tilde{L} = L = 1$.)

- ▶ We can read $\tilde{\mathbf{h}}$, \mathbf{h} from their Fourier series and build $\tilde{\mathbf{g}}$, and \mathbf{g} from them!
- ▶ $\tilde{\mathbf{h}}$ and \mathbf{h} are called **biorthogonal filters**.
- ▶ It can also be shown that there exist **symmetric** filters $\tilde{\mathbf{h}}$ and \mathbf{h} that satisfy (2).
- ▶ How do we find $\tilde{H}(\omega)$ and $H(\omega)$ that satisfy (2)?



- ▶ It can be shown that

$$\tilde{g}_k = (-1)^k h_{L-k} \quad \text{and} \quad g_k = (-1)^k \tilde{h}_{\tilde{L}-k}$$

where \tilde{L} and L are odd positive integers. (We typically take $\tilde{L} = L = 1$.)

- ▶ We can read $\tilde{\mathbf{h}}$, \mathbf{h} from their Fourier series and build $\tilde{\mathbf{g}}$, and \mathbf{g} from them!
- ▶ $\tilde{\mathbf{h}}$ and \mathbf{h} are called **biorthogonal filters**.
- ▶ It can also be shown that there exist **symmetric** filters $\tilde{\mathbf{h}}$ and \mathbf{h} that satisfy (2).
- ▶ How do we find $\tilde{H}(\omega)$ and $H(\omega)$ that satisfy (2)?



- ▶ It can be shown that

$$\tilde{g}_k = (-1)^k h_{L-k} \quad \text{and} \quad g_k = (-1)^k \tilde{h}_{\tilde{L}-k}$$

where \tilde{L} and L are odd positive integers. (We typically take $\tilde{L} = L = 1$.)

- ▶ We can read $\tilde{\mathbf{h}}$, \mathbf{h} from their Fourier series and build $\tilde{\mathbf{g}}$, and \mathbf{g} from them!
- ▶ $\tilde{\mathbf{h}}$ and \mathbf{h} are called **biorthogonal filters**.
- ▶ It can also be shown that there exist **symmetric** filters $\tilde{\mathbf{h}}$ and \mathbf{h} that satisfy (2).
- ▶ How do we find $\tilde{H}(\omega)$ and $H(\omega)$ that satisfy (2)?



- ▶ It can be shown that

$$\tilde{g}_k = (-1)^k h_{L-k} \quad \text{and} \quad g_k = (-1)^k \tilde{h}_{\tilde{L}-k}$$

where \tilde{L} and L are odd positive integers. (We typically take $\tilde{L} = L = 1$.)

- ▶ We can read $\tilde{\mathbf{h}}$, \mathbf{h} from their Fourier series and build $\tilde{\mathbf{g}}$, and \mathbf{g} from them!
- ▶ $\tilde{\mathbf{h}}$ and \mathbf{h} are called **biorthogonal filters**.
- ▶ It can also be shown that there exist **symmetric** filters $\tilde{\mathbf{h}}$ and \mathbf{h} that satisfy (2).
- ▶ How do we find $\tilde{H}(\omega)$ and $H(\omega)$ that satisfy (2)?



So how do we solve

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega + \pi)\overline{H(\omega + \pi)} = 2?$$

- ▶ Easy! Just pick $\tilde{\mathbf{h}}$ and then solve for \mathbf{h} .
- ▶ Let's pick $\tilde{\mathbf{h}}$ to be a symmetric, length-3, lowpass filter
 $\tilde{\mathbf{h}} = (\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1) = (\tilde{h}_1, \tilde{h}_0, \tilde{h}_1)$.
- ▶ So we are looking for two numbers \tilde{h}_0, \tilde{h}_1 so that

$$\tilde{h}_1 + \tilde{h}_0 + \tilde{h}_1 = \sqrt{2} \quad \tilde{h}_1 - \tilde{h}_0 + \tilde{h}_1 = 0$$

or

$$\begin{aligned} 2\tilde{h}_1 + \tilde{h}_0 &= \sqrt{2} \\ 2\tilde{h}_1 - \tilde{h}_0 &= 0 \end{aligned}$$



So how do we solve

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega + \pi)\overline{H(\omega + \pi)} = 2?$$

- ▶ Easy! Just pick $\tilde{\mathbf{h}}$ and then solve for \mathbf{h} .
- ▶ Let's pick $\tilde{\mathbf{h}}$ to be a symmetric, length-3, lowpass filter
 $\tilde{\mathbf{h}} = (\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1) = (\tilde{h}_1, \tilde{h}_0, \tilde{h}_1)$.
- ▶ So we are looking for two numbers \tilde{h}_0, \tilde{h}_1 so that

$$\tilde{h}_1 + \tilde{h}_0 + \tilde{h}_1 = \sqrt{2} \quad \tilde{h}_1 - \tilde{h}_0 + \tilde{h}_1 = 0$$

or

$$\begin{aligned} 2\tilde{h}_1 + \tilde{h}_0 &= \sqrt{2} \\ 2\tilde{h}_1 - \tilde{h}_0 &= 0 \end{aligned}$$



So how do we solve

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega + \pi)\overline{H(\omega + \pi)} = 2?$$

- ▶ Easy! Just pick $\tilde{\mathbf{h}}$ and then solve for \mathbf{h} .
- ▶ Let's pick $\tilde{\mathbf{h}}$ to be a symmetric, length-3, lowpass filter
 $\tilde{\mathbf{h}} = (\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1) = (\tilde{h}_1, \tilde{h}_0, \tilde{h}_1)$.
- ▶ So we are looking for two numbers \tilde{h}_0, \tilde{h}_1 so that

$$\tilde{h}_1 + \tilde{h}_0 + \tilde{h}_1 = \sqrt{2} \quad \tilde{h}_1 - \tilde{h}_0 + \tilde{h}_1 = 0$$

or

$$\begin{aligned} 2\tilde{h}_1 + \tilde{h}_0 &= \sqrt{2} \\ 2\tilde{h}_1 - \tilde{h}_0 &= 0 \end{aligned}$$



So how do we solve

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega + \pi)\overline{H(\omega + \pi)} = 2?$$

- ▶ Easy! Just pick $\tilde{\mathbf{h}}$ and then solve for \mathbf{h} .
- ▶ Let's pick $\tilde{\mathbf{h}}$ to be a symmetric, length-3, lowpass filter
 $\tilde{\mathbf{h}} = (\tilde{h}_{-1}, \tilde{h}_0, \tilde{h}_1) = (\tilde{h}_1, \tilde{h}_0, \tilde{h}_1)$.
- ▶ So we are looking for two numbers \tilde{h}_0, \tilde{h}_1 so that

$$\tilde{h}_1 + \tilde{h}_0 + \tilde{h}_1 = \sqrt{2} \quad \tilde{h}_1 - \tilde{h}_0 + \tilde{h}_1 = 0$$

or

$$\begin{aligned} 2\tilde{h}_1 + \tilde{h}_0 &= \sqrt{2} \\ 2\tilde{h}_1 - \tilde{h}_0 &= 0 \end{aligned}$$



The solution is

$$\tilde{h}_1 = \sqrt{2}/4 \quad \text{and} \quad \tilde{h}_0 = \sqrt{2}/2$$



- ▶ What about \mathbf{h} ?
- ▶ Let's let \mathbf{h} be length 3.

$$\begin{array}{ccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ h_1 & h_0 & h_1 \end{array}$$

doesn't work.

- ▶ Let's try

$$\begin{array}{ccccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & & \\ h_2 & h_1 & h_0 & h_1 & h_2 \end{array}$$

- ▶ Writing down the linear system satisfied by these orthogonality conditions and the linear constraint $H(\pi) = 0$ gives



- ▶ What about \mathbf{h} ?
- ▶ Let's let \mathbf{h} be length 3.

$$\begin{array}{ccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ h_1 & h_0 & h_1 \end{array}$$

doesn't work.

- ▶ Let's try

$$\begin{array}{ccccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & & \\ h_2 & h_1 & h_0 & h_1 & h_2 \end{array}$$

- ▶ Writing down the linear system satisfied by these orthogonality conditions and the linear constraint $H(\pi) = 0$ gives



- ▶ What about \mathbf{h} ?
- ▶ Let's let \mathbf{h} be length 3.

$$\begin{array}{ccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ h_1 & h_0 & h_1 \end{array}$$

doesn't work.

- ▶ Let's try

$$\begin{array}{ccccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & & \\ h_2 & h_1 & h_0 & h_1 & h_2 \end{array}$$

- ▶ Writing down the linear system satisfied by these orthogonality conditions and the linear constraint $H(\pi) = 0$ gives



- ▶ What about \mathbf{h} ?
- ▶ Let's let \mathbf{h} be length 3.

$$\begin{array}{ccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} \\ h_1 & h_0 & h_1 \end{array}$$

doesn't work.

- ▶ Let's try

$$\begin{array}{ccccc} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & & \\ h_2 & h_1 & h_0 & h_1 & h_2 \end{array}$$

- ▶ Writing down the linear system satisfied by these orthogonality conditions and the linear constraint $H(\pi) = 0$ gives



$$\begin{aligned}h_0 + h_1 &= \sqrt{2} \\h_1 + 2h_2 &= 0 \\h_0 - 2h_1 + 2h_2 &= 0\end{aligned}\tag{2}$$

The solution is $h_2 = -\frac{\sqrt{2}}{8}$, $h_1 = \frac{\sqrt{2}}{4}$, and $h_0 = \frac{3\sqrt{2}}{4}$ and the pair $\tilde{\mathbf{h}}$ and \mathbf{h} are called the **(5, 3) biorthogonal spline filter pair**.



We can build other filter pairs in this way. For \tilde{N} even, we take

$$\begin{aligned}\tilde{H}(\omega) &= \sqrt{2} \cos^{\tilde{N}}(\omega/2) \\ &= \frac{\sqrt{2}}{2^{\tilde{N}}} \sum_{k=-\tilde{N}/2}^{\tilde{N}/2} \binom{\tilde{N}}{\tilde{N}/2 - k} e^{ik\omega}\end{aligned}$$

The Fourier coefficients are from Pascal's triangle!! \tilde{h} is called the **spline filter**. There is an analogous construction for N and \tilde{N} both odd.



We can then pick a length for \mathbf{h} and solve the system. As the filter length grows, it is necessary to add derivative conditions to $H(\omega)$ at $\omega = \pi$. This is a bit tedious, so we instead turn to a result by Daubechies: There is an analogous construction for N and \tilde{N} both odd.



THEOREM (DAUBECHIES)

If $\tilde{N} = 2\tilde{\ell}$ and $N = 2\ell$, and $\tilde{H}(\omega)$ is the spline filter, then

$$H(\omega) = \sqrt{2} \cos^N(\omega/2) \sum_{j=0}^{\ell+\tilde{\ell}-1} \binom{\ell+\tilde{\ell}-1+j}{j} \sin^{2j}(\omega/2)$$

is a symmetric odd-length filter that satisfies $H(0) = \sqrt{2}$ and $H(\pi) = 0$ and $H(\omega)$, $\tilde{H}(\omega)$ satisfy the orthogonality condition (1).

There is an analogous construction for N and \tilde{N} both odd.

