

CODING THE HAAR WAVELET TRANSFORM

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PREP - Wavelet Workshop, 2009



TODAY'S SCHEDULE

- 9:00-9:15 **Welcome, Introductions**
- 9:15-10:15 **Lecture One:** Why Wavelets?
- 10:15-10:30 **Coffee Break** (SCA 202)
- 10:30-11:45 **Lecture Two:** Digital Images
- 12:00-1:00 **Lunch**
- 1:30-2:45 **Lecture Three:** The Discrete Haar Wavelet Transform
- 2:45-3:00 **Coffee Break** (SCA 202)
- 3:00-4:15 \Rightarrow **Lecture Four:** Coding the HWT, Edge Detection

OUTLINE

TODAY'S SCHEDULE

CODING THE HAAR TRANSFORM

Implementing W_N

Implementing W_N^T

Coding the 2D Transform

IN THE CLASSROOM

Teaching Ideas

Computer Usage

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

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CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

- ▶ The natural inclination when coding the HWT is to simply write a loop and compute the lowpass portion and the highpass portion in the same loop.
- ▶ This bogs down in Mathematica and is also difficult to generalize when we consider longer filters.
- ▶ If we look at the lowpass portion of the transform, $H\mathbf{v}$, we can see a better way to code things.

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

Consider $H\mathbf{v}$ when $\mathbf{v} \in \mathbb{R}^8$. We have

$$H\mathbf{v} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} v_1 + v_2 \\ v_3 + v_4 \\ v_5 + v_6 \\ v_7 + v_8 \end{bmatrix}$$

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

If we rewrite this, we have

$$H\mathbf{v} = \frac{1}{2} \begin{bmatrix} v_1 + v_2 \\ v_3 + v_4 \\ v_5 + v_6 \\ v_7 + v_8 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \\ v_5 & v_6 \\ v_7 & v_8 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \mathbf{v}\mathbf{h}$$

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

In a similar way we see that

$$G\mathbf{v} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -v_1 + v_2 \\ -v_3 + v_4 \\ -v_5 + v_6 \\ -v_7 + v_8 \end{bmatrix}$$

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

Rewriting gives

$$G\mathbf{v} = \frac{1}{2} \begin{bmatrix} -v_1 + v_2 \\ -v_3 + v_4 \\ -v_5 + v_6 \\ -v_7 + v_8 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \\ v_5 & v_6 \\ v_7 & v_8 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = V\mathbf{g}$$

So all we need to do to compute $W_n\mathbf{v}$ is to create V , multiply it with \mathbf{h} and \mathbf{g} , and join to the two blocks together!

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

Here is some Matlab code to do it:

```
function y=MyHWT1D1(v)
    V=reshape(v,2,length(v)/2)';
    lp=V*[1/2,1/2];
    hp=V*[-1/2,1/2];
    y=[lp;hp];
```

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N

And some Mathematica code as well:

```
MyHWT1D1[v_] := Module[{V, lp, hp, y},  
  V = Partition[v, 2];  
  lp = V.{1/2, 1/2};  
  hp = V.{-1/2, 1/2};  
  y = Join[lp, hp];  
  Return[y];  
];
```

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

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- ▶ Let's again look at a vector $\mathbf{v} \in \mathbb{R}^8$ and consider the product $W_8^T \mathbf{v}$:

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

$$W_8^T \mathbf{v} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix}$$

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

or

$$W_8^T \mathbf{v} = \begin{bmatrix} v_1 - v_5 \\ v_1 + v_5 \\ v_2 - v_6 \\ v_2 + v_6 \\ v_3 - v_7 \\ v_3 + v_7 \\ v_4 - v_8 \\ v_4 + v_8 \end{bmatrix}$$

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

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- ▶ We need to dot V with both $\mathbf{h} = [1, 1]^T$ and $\mathbf{g} = [1, -1]^T$ but then intertwine the results.

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

Given $\mathbf{v} = (v_1, \dots, v_8)^T$, how can we use Matlab/Mathematica commands to form

$$V = \begin{bmatrix} v_1 & v_5 \\ v_2 & v_6 \\ v_3 & v_7 \\ v_4 & v_8 \end{bmatrix}$$

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

Answer (Matlab):

```
V = reshape(v, length(v)/2, 2);
```

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

Answer (Mathematica):

```
V = Transpose[Partition[v, Length[v]/2]];
```

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

We can compute

$$\mathbf{Vh} = \begin{bmatrix} v_1 + v_5 \\ v_2 + v_6 \\ v_3 + v_7 \\ v_4 + v_8 \end{bmatrix} \quad \text{and} \quad \mathbf{Vg} = \begin{bmatrix} v_1 - v_5 \\ v_2 - v_6 \\ v_3 - v_7 \\ v_4 - v_8 \end{bmatrix}$$

How do we form $W_8^T \mathbf{v}$ from this?

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

Answer (Matlab):

```
reshape([V*g V*h]',1,length(v))';
```

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

Answer (Mathematica):

```
Flatten[Transpose[{ V g, V h } ]];
```

or one I just learned:

```
Riffle[ Vg, Vh ];
```

CODING THE HAAR TRANSFORM

IMPLEMENTING W_N^T

Let's have a look at the Mathematica notebook

```
HaarTransform1D.nb
```

to see how to code the one-dimensional HWT and its inverse.

2D HAAR TRANSFORM

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- ▶ We can easily write a routine to compute $C = W_N A$. Let's call this routine \bar{w} .
- ▶ Our goal is to compute $B = W_N A W_N^T = C W_N^T$.
- ▶ It turns out that writing code for $C W_N^T$ is a bit tedious, but if we use some linear algebra ...

2D HAAR TRANSFORM

CODING THE 2D TRANSFORM

- ▶ If we transpose both sides of $B = CW_N^T$, we have

$$B^T = W_N C^T$$

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CODING THE 2D TRANSFORM

- ▶ If we transpose both sides of $B = CW_N^T$, we have

$$B^T = W_N C^T$$

- ▶ So we can simply apply W to C^T and transpose the result.
- ▶ One student wasn't so sure about this . . .

2D HAAR TRANSFORM

CODING THE 2D TRANSFORM

IN THE CLASSROOM

TEACHING IDEAS

- ▶ The students really enjoy the material in this chapter. It is quite straightforward and ties together everything new we've done to date.

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TEACHING IDEAS

- ▶ The students really enjoy the material in this chapter. It is quite straightforward and ties together everything new we've done to date.
- ▶ We have them look at the entropy of particular vectors when processed by the Haar transform. This gives them some idea of the potential for wavelet-based compression.

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IN THE CLASSROOM

COMPUTER USAGE

- ▶ As you might imagine, we do lots of coding in this chapter.
- ▶ We let the students work in pairs and they write code for the Haar transform and its inverse (1D and 2D) as well as iterated versions.
- ▶ They can get pretty frustrated with Mathematica at this point - it is good to show them some simple debugging techniques.

IN THE CLASSROOM

COMPUTER USAGE

- ▶ We let them use their own images/audio files (sometimes dangerous).

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COMPUTER USAGE

- ▶ We let them use their own images/audio files (sometimes dangerous).
- ▶ To test their iterated 1D inverse, they must download an audio clip from the course web site that has been transformed p times, guess at p , and then apply their inverse to guess the movie clip.

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⇒Dinner and free time