Daubechies Filters

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PREP - Wavelet Workshop, 2009

TODAY'S SCHEDULE

9:00-10:15	**Lecture Five:** Cumulative Energy, Quantization, and Peak Signal–to–Noise Ratio
10:15-10:30	Coffee Break (SCA 202)
10:30-11:45	**Lecture Six:** Huffman Coding
12:00-1:00	Lunch
1:30-2:45	**Lecture Seven:** Putting it All Together: Image Compression
2:45-3:00	Coffee Break (SCA 202)
3:00-4:15	⇒**Lecture Eight:** Daubechies Wavelet Transformations

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Outline

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DAUBECHIES D4 TRANSFORMATION

WHY LONGER FILTERS?

- We were able to use the Haar Wavelet Transformation in applications, but there are some drawbacks.

DAUBECHIES D4 TRANSFORMATION

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Daubechies D4 Transformation
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- The Haar transform of this vector is

$$\sqrt{2}\,(0, 100 \mid 0, 0)$$

Daubechies D4 Transformation
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- If we are looking for edges, Haar fails.

Daubechies D4 Transformation

Why Longer Filters?

- The problem is the Haar filter

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Daubechies D4 Transformation
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Daubechies D4 Transformation
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- In addition, we want to preserve the matrix structure and its properties.

Daubechies D4 Transformation
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- In this lecture we will look to build longer filters that generate orthogonal transforms.
- In addition, we want to preserve the matrix structure and its properties.
- What about a length 3 filter?

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Daubechies D4 Transformation
Why Longer Filters?

Let's write out the top portion of a wavelet transform matrix using the filter (h_0, h_1, h_2) applied to an 6-vector:

$$H\mathbf{v} = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 \\ h_0 & 0 & 0 & 0 & h_2 & h_1 \end{bmatrix} \cdot \mathbf{v}$$

There are two things to consider:

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Daubechies D4 Transformation
Why Longer Filters?

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Daubechies D4 Transformation

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Daubechies D4 Transformation

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- The downside is wrapping implicitly assumes the data are periodic.
- The other problem is the orthogonality conditions. If the rows of H are going to be orthonormal, we need:

DAUBECHIES D4 TRANSFORMATION

WHY LONGER FILTERS?

$$H = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & 0 \\ h_0 & 0 & 0 & 0 & h_2 & h_1 \end{bmatrix} \cdot \begin{bmatrix} h_2 & 0 & h_0 \\ h_1 & 0 & 0 \\ h_0 & h_2 & 0 \\ 0 & h_1 & 0 \\ 0 & h_0 & h_2 \\ 0 & 0 & h_1 \end{bmatrix} = I_3$$

$$\begin{aligned} h_0^2 + h_1^2 + h_2^2 &= 1 \\ h_0 h_2 &= 0 \end{aligned}$$

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Daubechies D4 Transformation

Why Longer Filters?

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DAUBECHIES D4 TRANSFORMATION
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- It is pretty easy to verify that odd-length filters all have this problem.

DAUBECHIES D4 TRANSFORMATION
WHY LONGER FILTERS?

- The condition $h_0 h_2 = 0$ implies either $h_0 = 0$ or $h_2 = 0$ so we're back to a length 2 filter!
- It is pretty easy to verify that odd-length filters all have this problem.
- The smallest even length filter bigger than 2 is 4 - let's look at that one.

DAUBECHIES LENGTH FOUR FILTER

DAUBECHIES SYSTEM

Let's start by writing the top half of the wavelet matrix with filter $\mathbf{h} = (h_0, h_1, h_2, h_3)$ that we would use to process a length 8 vector:

$$H = \begin{bmatrix} h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 \\ h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 \end{bmatrix}$$

The orthogonality conditions are:

Daubechies Length Four Filter
Daubechies System

- $$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\ h_0 h_2 + h_1 h_3 &= 0 \end{aligned}$$

Daubechies Length Four Filter
Daubechies System

- $$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\ h_0 h_2 + h_1 h_3 &= 0 \end{aligned}$$
- Suppose we also want **h** to satisfy properties similar to that of the Haar transform, that is, preserve constant data and “kill" oscillatory data:
$$h_0 - h_1 + h_2 - h_3 = 0$$

Daubechies Length Four Filter

Daubechies System

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- Suppose we also want **h** to satisfy properties similar to that of the Haar transform, that is, preserve constant data and "kill" oscillatory data:
  $$h_0 - h_1 + h_2 - h_3 = 0$$
- We do get one "freebie" out of these conditions. It turns out that if these conditions are satisfied, then
  $$h_0 + h_1 + h_2 + h_3 = \pm\sqrt{2}$$
  (Problem 7.2, Page 243).

Daubechies Length Four Filter

Daubechies System

Let's pick the positive root. Then our system becomes

$$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\ h_0 h_2 + h_1 h_3 &= 0 \\ h_0 + h_1 + h_2 + h_3 &= \sqrt{2} \\ h_0 - h_1 + h_2 - h_3 &= 0 \end{aligned}$$

Daubechies Length Four Filter
Daubechies System

- It is straightforward to show that there is an infinite number of solutions to this system and we haven't even considered the bottom half of the matrix yet.

Daubechies Length Four Filter
Daubechies System

- It is straightforward to show that there is an infinite number of solutions to this system and we haven't even considered the bottom half of the matrix yet.
- If we write out the bottom half of the wavelet transform matrix, we have:

DAUBECHIES LENGTH FOUR FILTER
DAUBECHIES SYSTEM

$$G = \begin{bmatrix} g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 \\ g_1 & g_0 & 0 & 0 & 0 & 0 & g_3 & g_2 \end{bmatrix}$$

and the orthogonality constraints for $\mathbf{g}$ are the same as $\mathbf{h}$:

$$\begin{aligned} g_0^2 + g_1^2 + g_2^2 + g_3^2 &= 1 \\ g_0 g_2 + g_1 g_3 &= 0 \end{aligned}$$

Daubechies Length Four Filter
Daubechies System

Let's write out the *entire* matrix and look at conditions that must be satisfied in order for W_8 to be orthogonal. We have:

DAUBECHIES LENGTH FOUR FILTER
DAUBECHIES SYSTEM

$$W = \begin{bmatrix} h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 \\ h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 \\ \hline g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 \\ g_1 & g_0 & 0 & 0 & 0 & 0 & g_3 & g_2 \end{bmatrix}$$

Daubechies Length Four Filter
Daubechies System

Note that the vectors $\mathbf{h} = (h_0, h_1, h_2, h_3)$ and $\mathbf{g} = (h_3, -h_2, h_1, -h_0)$ are orthogonal!

Daubechies Length Four Filter
Daubechies System

Moreover, whenever

$$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\ h_0 h_2 + h_1 h_3 &= 0 \end{aligned}$$

we have

$$\begin{aligned} g_0^2 + g_1^2 + g_2^2 + g_3^2 = h_3^2 + (-h_2)^2 + h_1^2 + (-h_0)^2 &= 1 \\ g_0 g_2 + g_1 g_3 = h_3 h_1 + (-h_2)(-h_0) &= 0 \end{aligned}$$

Daubechies Length Four Filter
Daubechies System

- so the bottom half of the matrix consists of orthonormal rows.

Daubechies Length Four Filter
Daubechies System

- so the bottom half of the matrix consists of orthonormal rows.
- You can also check the other orthogonality conditions, and they all hold.

Daubechies Length Four Filter
Daubechies System

To summarize, if we choose $\mathbf{h} = (h_0, h_1, h_2, h_3)$ so that

$$\begin{aligned} h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\ h_0 h_2 + h_1 h_3 &= 0 \\ h_0 + h_1 + h_2 + h_3 &= \sqrt{2} \\ h_0 - h_1 + h_2 - h_3 &= 0 \end{aligned}$$

and $\mathbf{g} = (h_3, -h_2, h_1, -h_0)$, then the matrix W_N is orthogonal! There's just one small problem ...

Daubechies Length Four Filter
Daubechies System

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Daubechies Length Four Filter
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DAUBECHIES SYSTEM

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- As we tell out students, engineers see this as an opportunity and add another condition!
- The condition Ingrid Daubechies added was incredibly simple:

  Why not force the high pass portion of the matrix to kill linear signals?

Daubechies Length Four Filter
Daubechies System

- The system has an infinite number of solutions!
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  Why not force the high pass portion of the matrix to kill linear signals?

- That is, add the condition

$$h_1 - 2h_2 + 3h_3 = 0$$

Daubechies Length Four Filter
Daubechies System

Our system becomes

$$
\begin{aligned}
h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\
h_0 h_2 + h_1 h_3 &= 0 \\
h_0 + h_1 + h_2 + h_3 &= \sqrt{2} \\
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\end{aligned}
$$

It turns out that there are two real solutions to this system.

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DAUBECHIES LENGTH FOUR FILTER
SOLVING THE SYSTEM

Let's solve this system. The equation

$$h_0 h_2 + h_1 h_3 = 0$$

is equivalent to saying the vectors (h_0, h_1) and (h_2, h_3) are orthogonal. This implies that for some $c \neq 0$,

$$(h_2, h_3) = c(-h_1, h_0)$$

Plugging this into $h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1$ and simplifying gives

Daubechies Length Four Filter
Solving the System

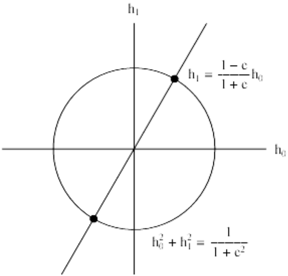
$$h_0^2 + h_1^2 = \frac{1}{1+c^2}$$

If we also plug $(h_2, h_3) = c(-h_1, h_0)$ into $h_0 - h_1 + h_2 - h_3 = 0$ and simplify, we have

$$h_1 = \left(\frac{1-c}{1+c}\right) h_0$$

DAUBECHIES LENGTH FOUR FILTER
SOLVING THE SYSTEM

Geometrically, we have the following graph:



Daubechies Length Four Filter

Solving the System

If we plug $(h_2, h_3) = c(-h_1, h_0)$ into

$$h_1 - 2h_2 + 3h_3 = 0$$

we obtain

$$h_1 = -\left(\frac{3c}{1+2c}\right) h_0$$

and this is another line through the origin. The only way we can have a solution is if the two lines through the origin have the same slope. That is,

Daubechies Length Four Filter
Solving the System

$$\frac{1-c}{1+c} = -\frac{3c}{1+2c}$$

or

$$c^2 + 4c + 1 = 0$$

so that $c = -2 \pm \sqrt{3}$.

Daubechies Length Four Filter
Solving the System

- Can you take the positive root $c = -2 + \sqrt{3}$ and find a solution to the system?

Daubechies Length Four Filter

Solving the System

- Can you take the positive root $c = -2 + \sqrt{3}$ and find a solution to the system?
- To let you know if you are on the right track, you should soon arrive at $h_1 = \sqrt{3}h_0$ and with a bit more work,

$$h_0 = \pm \frac{1 + \sqrt{3}}{4\sqrt{2}}$$

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DAUBECHIES LENGTH FOUR FILTER
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- Take the positive value and proceed.
- OR ... can you plug the system into Mathematica/Matlab and solve it?

DAUBECHIES LENGTH FOUR FILTER

SOLVING THE SYSTEM

The solution is

$$h_0 = \frac{1}{4\sqrt{2}}(1+\sqrt{3}) \qquad h_1 = \frac{1}{4\sqrt{2}}(3+\sqrt{3})$$

$$h_2 = \frac{1}{4\sqrt{2}}(3-\sqrt{3}) \qquad h_3 = \frac{1}{4\sqrt{2}}(1-\sqrt{3})$$

The solution with the negative root $c = -2-\sqrt{3}$ is the reflection of the above solution.

Daubechies 6-Term Filter
The System

- If we repeat the process with $\mathbf{h} = (h_0, \ldots, h_5)$, we arrive at the following orthogonality conditions:

$$\begin{aligned} \sum_{k=0}^{5} h_k^2 &= 1 \\ h_0 h_2 + h_1 h_3 + h_2 h_4 + h_3 h_5 &= 0 \\ h_0 h_4 + h_1 h_5 &= 0 \end{aligned}$$

DAUBECHIES 6-TERM FILTER
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- We have the two additional conditions:

$$\begin{aligned} h_0 + h_1 + h_2 + h_3 + h_4 + h_5 &= \sqrt{2} \\ h_0 - h_1 + h_2 - h_3 + h_4 - h_5 &= 0 \end{aligned}$$

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DAUBECHIES 6-TERM FILTER
THE SYSTEM

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DAUBECHIES 6-TERM FILTER

THE SYSTEM

- If we pick $\mathbf{g} = (h_5, -h_4, h_3, -h_2, h_1, -h_0)$, then W_N is an orthogonal wavelet matrix.
- This time, we need to add two conditions, ensuring that the high pass portion of the matrix kills linear and quadratic signals:

$$\begin{aligned} h_1 - 2h_2 + 3h_3 - 4h_4 + 5h_5 &= 0 \\ h_1 - 4h_2 + 9h_3 - 16h_4 + 25h_5 &= 0 \end{aligned}$$

DAUBECHIES 6-TERM FILTER
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- We get the following system:

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DAUBECHIES 6-TERM FILTER
THE SYSTEM

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Daubechies 6-Term Filter
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- It turns out there are two real solutions to this system although it is *really hard* to solve it by hand (see Problem 7.14 Page 248).

DAUBECHIES 6-TERM FILTER
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DAUBECHIES 6-TERM FILTER
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- You can prove the other solution is

$$(h_5, h_4, h_3, h_2, h_1, h_0)$$

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DAUBECHIES 6-TERM FILTER
THE SYSTEM

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- Longer filters result in systems that must be solved numerically.

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Daubechies Filters of Even Length

- In general, suppose $\mathbf{h} = (h_0, \ldots, h_L)$ where L is an even positive integer.

Daubechies Filters of Even Length

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- Daubechies proved that the general system has $2^{\lfloor \frac{L+2}{4} \rfloor}$ real solutions.

DAUBECHIES FILTERS OF EVEN LENGTH

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- Daubechies proved that the general system has $2^{\lfloor \frac{L+2}{4} \rfloor}$ real solutions.
- She also proved that if $(h_0, \ldots, h_L)$ is a solution, so is $(h_L, \ldots, h_0)$.

Daubechies Filters of Even Length

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- Daubechies proved that the general system has $2^{\lfloor \frac{L+2}{4} \rfloor}$ real solutions.
- She also proved that if $(h_0, \ldots, h_L)$ is a solution, so is $(h_L, \ldots, h_0)$.
- Daubechies suggests a method for finding her so-called *extremal phase* filters.

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