Optimal Routing of a Sailboat in Steady Winds

Nathan Greiner (Class of 2008)
Advisor: Dr. Mike Hennessey
Center for Applied Mathematics
University of St. Thomas
September 2005

Abstract

From crossing an ocean to modern day sailboat races, people have always been looking for the quickest way to get from one point to another. By applying concepts of calculus of variations and a numerical ordinary differential equation solver, this research project strives to find an optimal route for a sailboat through a steady wind field with constant direction but varying speed.

This project is a continuation of a research project Dr. Hennessey took on where he attempted to find the optimal route of a sailboat through a steady wind field with the speed of the wind being constant and the direction varying with position.

The dual simulation to Dr. Hennessey's previous work (this project) pertains to a sailboat traveling down the center of a channel with steady winds, and this sailboat wants to get to a point on the shore down stream in the minimal time possible. As in many fluid mechanics problems, the flow of a fluid is slower at the edges than in the center, as is the case in this simulation. I will assume that there is a linear decrease in the speed of the wind in the center of the channel to a smaller speed at the shore of the channel. This project will apply many different methods to attempt to find the minimum time route including simple integrals, Using derivatives to find a minimum, and finally using calculus of variations to find the optimal smooth curve through the given wind field.

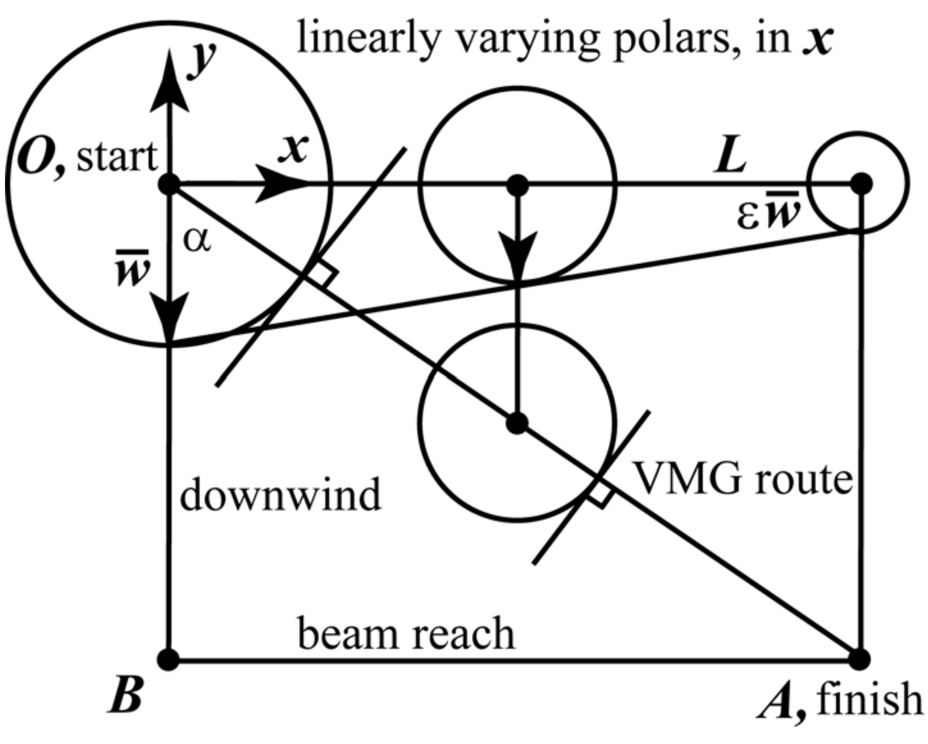
Introduction:

Consider a sailboat traveling down the center of a channel that is consistently 2 nautical miles in width and starts at the origin, point O. This boat wants to travel from the center of the channel to a point A on the shore 1 nautical mile down the channel from point O. This paper is interested in optimizing the time it takes to go from point O to point A.

In order to describe the optimal route, the variables of the problem must be described. In the interest of simplicity the effects of current will be ignored. For a typical channel flow problem in fluid mechanics, a fluid travels down the center of a channel the fastest and the speed decreases quadratically the closer it is to the edge of the channel. For this problem the speed of the fluid will linearly decrease from the center to the shore of the channel. The wind will therefore only change speed with respect to the x-axis and will always point straight down the channel.

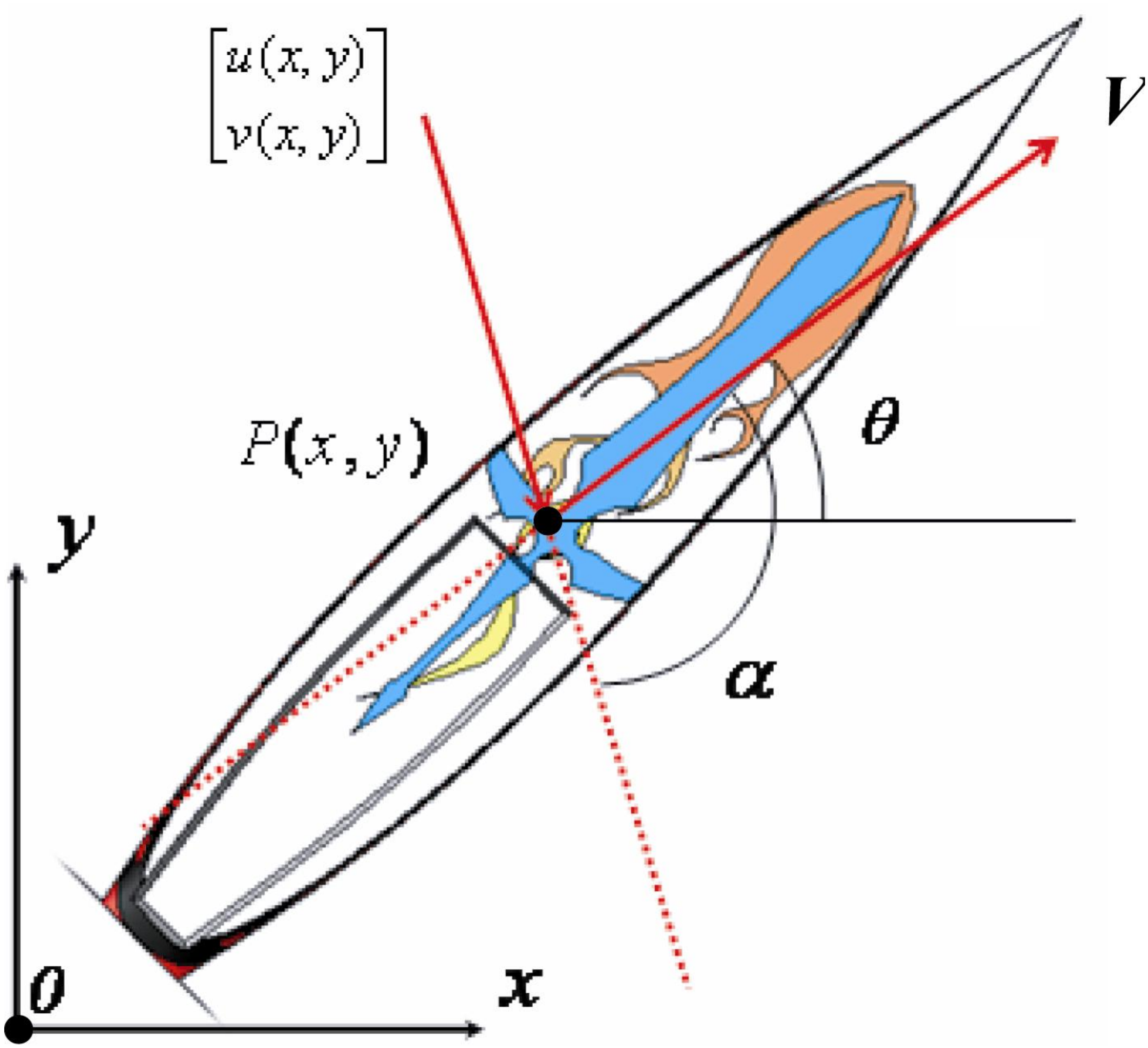
Initially, for the sake of simplicity, the boat will be able to travel the same speed any direction with what is known as an isotropic polar. Realistically there are inefficiencies a sailboat suffers when traveling in different directions with respect to the wind direction. This is called a boat polar and will be described further on in the paper.

Below is the graphical explanation for the problem:



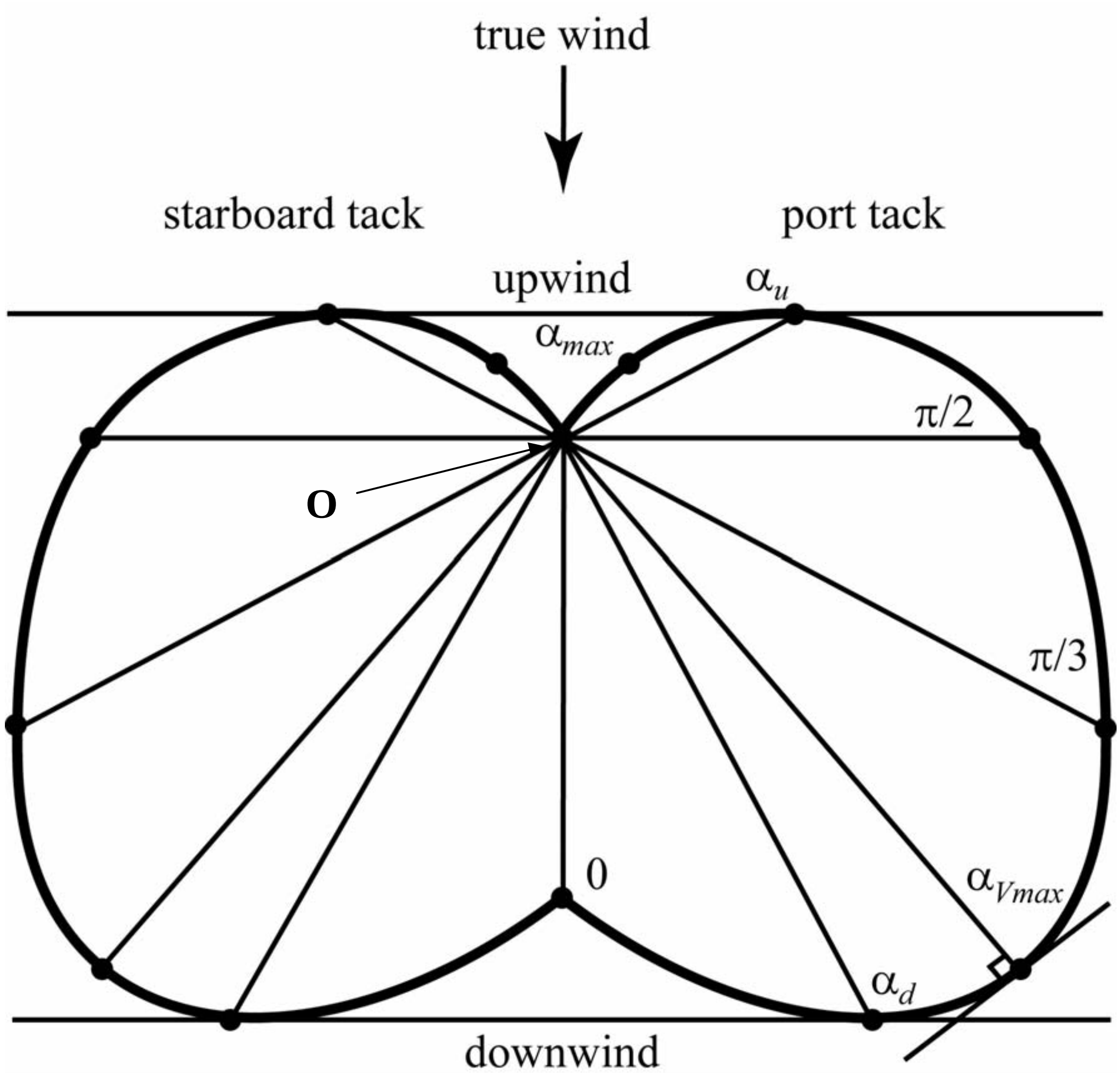
Preliminary Information:

1. Coordinate System



V is the speed of the boat; u and v make the x and y components of the wind velocity respectively; alpha is the dot product angle between the boat velocity and the wind vector; theta is the angle the boat velocity makes with the x axis; P is the location of the boat in x and y.

2. Realistic Boat Polar and VMG concept



Boat Polar: If a sailboat were placed at the origin (pt. O) and headed in any direction (given by alpha) the speed of the boat would be given by the length of the line leaving point O and intersecting the polar at the specific alpha.

VMG: VMG stands for velocity made good. Consider that a sailboater wants to go to a point directly down wind. The boater has the choice of going straight down wind, or going just off down wind (as represented by α_d) and jibing back to intersect the desired downwind point. It turns out that taking the off down wind approach (α_d) is quicker because it has a higher down wind component than if the boater just headed straight down wind.

Research Goals:

1. Find optimal route with isotropic polar
    a. Find time for bounding routes: OBA and OA
    b. Find optimal time for straight two legged route
    c. Find time optimal route using Calculus of Variations
2. Find optimal route with realistic boat polar
    a. Find time for bounding routes: OBA and OA
    b. Find time optimal route using Calculus of Variations

Results and Methodology:

1. Isotropic Polar

A. Bounding Routes

Finding the time for the bounding routes OBA and OA are just simple integrals:

$$T_{OBA} = \ldots \; x = 0.831 \text{ Hrs}$$

$$T_{OA} = \int_0 \ldots w \ldots \; x = 0.862 \text{ Hrs}$$

For this specific simulation: *wbar = 5.0 Knots; ε = 0.05; L = 1 NM (Nautical Mile);* $\alpha = \pi / 2$

w(x) is the speed of the wind with respect to x

When these values are put into the above integrals, they give:

T_{OBA} = 0.831 Hours
T_{OA} = 0.862 Hours

It is interesting and important to see that in this case the VMG route OA (which is also the optimal distance route) is not time optimal.

B. Optimal Two Legged Straight Line Route

Since the time optimal is not necessarily the distance optimal, it would be advantageous to try and balance varying distance and time by varying the initial theta and optimizing using a first derivative test.

The two legged route would consist of an initial leg extending from O to a point C exactly half way

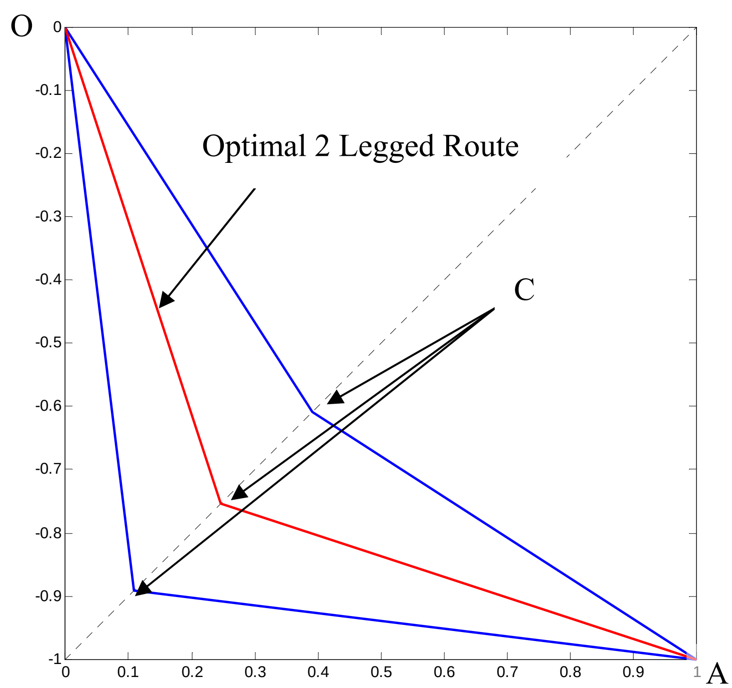
After many derivations to determine the limits of the integrals, optimizing the time can be reduced to a function of theta and the known terms of alpha and w(x), which is given as follows:

$T_{OCA} =$ [illegible]

When time is minimized using a Mathmatica minimization operation:

$T^{Opt}_{OCA} = 0.785$ Hours

This is still closer to time optimal than either of the bounding routes, but is not necessarily the fastest way to travel from point O to A.



C. Calculus of Variations Solution:

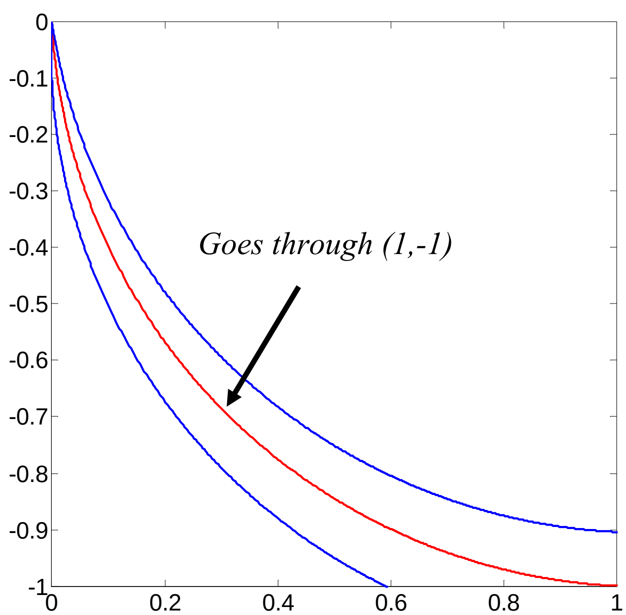
The mathematics behind the Calculus of variations is a very arduous and complicated process and is out of the scope of this paper. Suffice it to say the background mathematics is outlined in a book by Bryson and Ho, and their work has been used as a basis for this portion of the paper.

The Calculus of variations is used to find smooth optimal curves, shapes, control efforts, trajectories, etc. Since this paper is interested in finding the optimal route through the given wind field calculus of variations will be used to find the optimal smooth curve.

The calculus of variations derivation for this problem gives a fairly simple equation for the yaw rate (given by theta dot) of the sailboat on its passage from O to A. The equation is given by

$$\frac{d\theta}{dt} = -\frac{dU}{dx}\frac{dV}{dU}\,\text{Cos}\,\theta,\theta$$ [redacted]

with the initial theta being iterated on to run the trajectory through the finish point. This is a technique known as the shooting method. The shooting method is used because the initial x and y are known and the final x and y are known but the initial theta is not, so the shooting method is used to guess on an initial theta to force the result to go through the final x and y.



Using the shooting method, the final x and y will rarely be exactly (1,-1) but a final x and y can be reached so as to be "close enough" to the desired x and y.

Simulation and Results:

Using a numerical ordinary differential equation solver to solve the following differential equation:

$$\frac{dx}{dt} = V \text{ [redacted]}$$

$$\frac{dy}{dt} = V \text{ [redacted]}$$

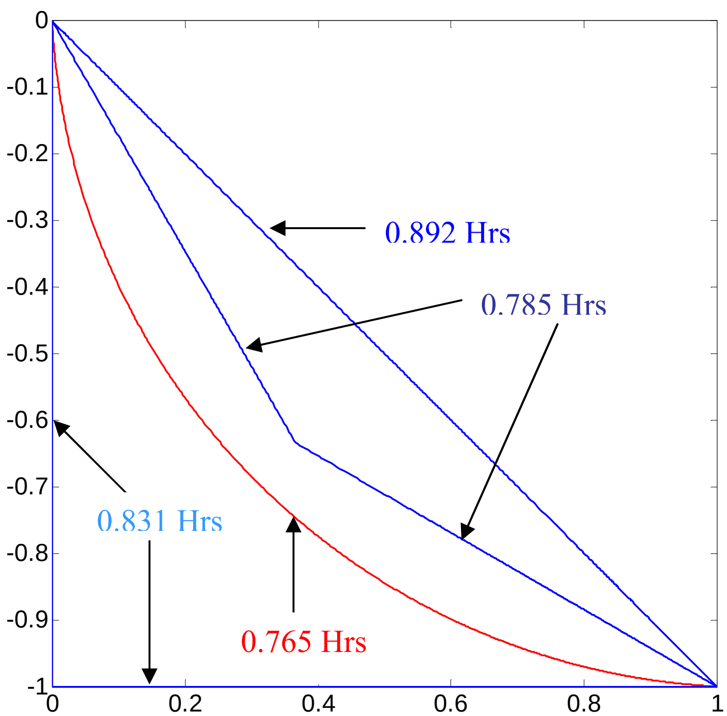
$$\frac{dq}{dt} = \frac{dV}{dU}\frac{dU}{dx} \text{Sin } q$$

and after iterating on the initial theta, the simulation returned the following results:

$x_f = 0.9999$, $y_f = -0.9999$, $t_f = 0.765$ Hours

The final x and y are determined to be sufficiently close to (1,-1) to say the initial theta was correct or "close enough". This said it is easy to see the final time for the calculus of variations is considerably faster than any of the previous results (T_{OBA}, T_{OA}, and T_{OCA}).

Isotropic Boat Polar Results



2. Realistic Boat Polar

A. Bounding Routes

The techniques for finding the bounding route times are the same as described in the isotropic case, except with the updated boat polar. The realistic boat polar V(x) can be described by

$$\frac{U}{U_{ref}} \quad \sum_{n=0} c_n \alpha^n$$

where U = w(x) (as previously defined); U_{ref} = 16.65 knots; c_0 = 4.8401; c_1 = 3.0336; c_2 = -4.7994; c_3 = 0.9402; c_4 = 1.0479; c_5 = -0.3765

w(x) is defined above

U_{ref} and c_0 through c_5 were all determined experimentally

The realistic boat polar will be used for the boundary integrals like in the isotropic polar case in order to get an idea of the optimal route time.

T_{OBA} = [illegible] = 3.[illegible]0 Hours

T_{OA} = [illegible] = 2[illegible] Hours

This is controversial to the result in the isotropic case when the VMG route (OA) took longer than the outer boundary route (OBA) since $T_{OBA} > T_{OA}$. As shown by the boat polar (page 4), it is much quicker to take the VMG route verses the downwind and beam reach approach because it is so inefficient to sail directly down wind.

The calculus of variations derivation for the realistic boat polar is much more complicated than the isotropic case and is included in Appendix A. The formula for theta dot is

$$[illegible] \frac{\partial^2 V}{\partial \alpha^2} V - \frac{2 \, [illegible] \frac{\partial V}{\partial \alpha}}{V} \frac{d\theta}{dt} = \frac{dU}{dx} [illegible] \frac{\partial V}{\partial \alpha} - V \frac{\partial^2 V}{\partial U \partial \alpha} [illegible] \theta - V \frac{\partial V}{\partial U} \sin\theta$$

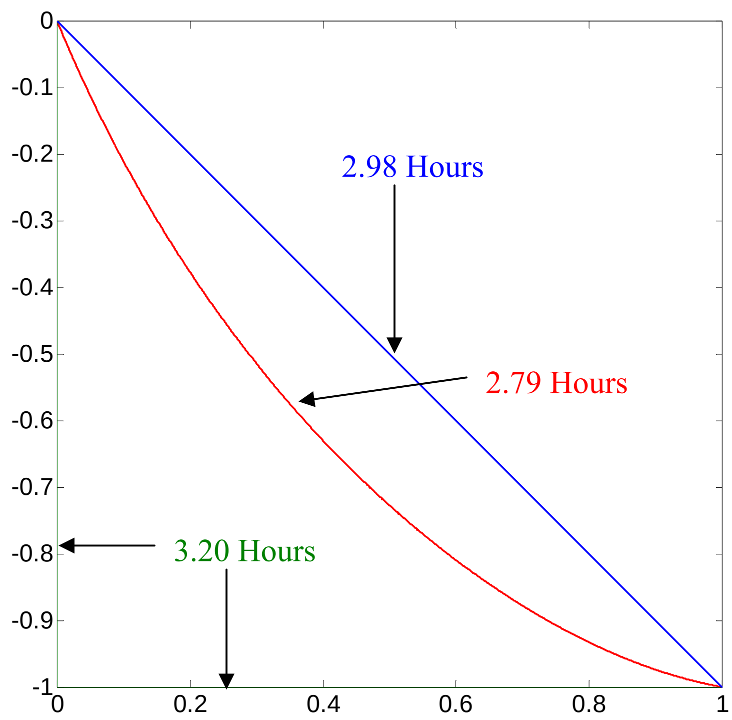
Simulation and Results:

When this is put into the Matlab numerical ordinary differential equation solver and the shooting method is used to find the initial theta, the final position and time is

$x_f = 1.0000$, $y_f = -1.0000$, $t_f = 2.7680$ Hours

This route is sufficiently shorter in time than either of the bounding routes and is assumed to be the optimal route from O to A.

Realistic Boat Polar Results



Future Direction:

1. Look at the sensitivity of Epsilon
2. Make the variation of wind non-linear (possibly use quadratic)
3. Adding in the effects of current
4. Both wind speed and direction variable

Acknowledgements:

1) J-term 2005 Students for following figures
    a. Linearly Varying Polar (Page 2)
    b. Sailboat Coordinate System (Page 3)
    c. Boat Polar (Page 4)

2) A.E. Bryson Jr. and Y.-C. Ho, Applied Optimal Control, Hemisphere, New York, 1975.

3) M. P. Hennessey, Jeffrey A. Jalkio, Christopher S. Greene, and Colin M. Sullivan, Optimal Routing of a Sailboat in Steady Winds, School of Engineering and the Center for Applied Mathematics, University of St. Thomas, December, 2005

Appendix A: $\dot{\theta}$ *Derivation*

Polar Family Formulation

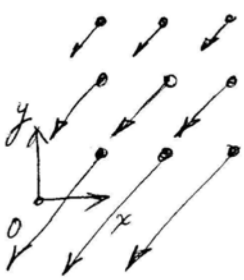
$$\dot{x} = V\left(V_w(y), \alpha(\theta)\right)\cos\theta$$

$$\dot{y} = V\left(V_w(y), \alpha(\theta)\right)\sin\theta$$

↑ wind speed

↑ wind angle only as a function of θ if we assume steady, straight, but spatially variable speed winds ⇒ polar family (e.g. in y direction)

[or (x, y) or x dependence as well]



$$H = \lambda_x V\left(V_w(y), \alpha(\theta)\right)\cos\theta + \lambda_y V\left(V_w(y), \alpha(\theta)\right)\sin\theta + 1$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x}\ ; \quad \dot{\lambda}_y = -\frac{\partial H}{\partial y}\ ; \quad 0 = \frac{\partial H}{\partial \theta}\ ; \quad H = 0$$

$$\frac{\partial H}{\partial \theta} = \lambda_x V\left(V_w(y), \alpha(\theta)\right)(-\sin\theta) + \lambda_x \cos\theta \frac{\partial V}{\partial \theta} +$$

$$\lambda_y V\left(V_w(y), \alpha(\theta)\right)(\cos\theta) + \lambda_y \sin\theta \frac{\partial V}{\partial \theta}$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}$$

So,
$$\frac{\partial H}{\partial \theta} = \frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\left(\lambda_x \cos\theta + \lambda_y \sin\theta\right) + V\left(-\lambda_x \sin\theta + \lambda_y \cos\theta\right) = 0$$

$H = 0$

$$\lambda_x V\cos\theta + \lambda_y V\sin\theta + 1 = 0$$

As before,

$$\begin{bmatrix} \frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\cos\theta - V\sin\theta & \left(\frac{\partial V}{\partial \alpha}\right)\frac{d\alpha}{d\theta}\sin\theta + V\cos\theta \\ V\cos\theta & V\sin\theta \end{bmatrix}\begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Solve for λ_x, λ_y

$$\lambda_x = \frac{-\left(\frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\sin\theta + V\cos\theta\right)}{V^2}$$

$$\lambda_y = \frac{\left(\frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\cos\theta - V\sin\theta\right)}{V^2}$$

$$\dot{\lambda}_y = \frac{\partial H}{\partial y} = 0$$

$$\dot{\lambda}_y = \frac{V^2 \frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\cos\theta - V\sin\theta\right) - \left(\frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta} - V\sin\theta\right) 2\frac{dV}{dt}}{V^4} = 0$$

$$V\frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\cos\theta - V\sin\theta\right) - \left(\frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\cos\theta - V\sin\theta\right) 2\frac{dV}{dt} = 0$$

($\frac{d\alpha}{d\theta} \to 1$)

$$\frac{dV}{dt}\left(U(x), \alpha(\theta)\right) = \frac{\partial V}{\partial U}\frac{dU}{dx}\frac{dx}{dt} + \frac{\partial V}{\partial \alpha}\frac{d\alpha}{d\theta}\frac{d\theta}{dt}$$

($\frac{dx}{dt} \to V\cos\theta$, $\frac{d\alpha}{d\theta} \to 1$)

So, $\frac{dV}{dt} = \frac{\partial V}{\partial U}\frac{dU}{dx}V\cos\theta + \frac{\partial V}{\partial \alpha}\dot{\theta}$

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\cos\theta - V\sin\theta\right) = \frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\cos\theta\right) - \frac{d}{dt}\left(V\sin\theta\right)$$

$$\frac{d}{dt}\left(V\sin\theta\right) = \left(\frac{dV}{dt}\sin\theta + V\cos(\theta)\dot{\theta}\right) = \left[\left(\frac{\partial V}{\partial U}\frac{dU}{dx}V\cos\theta + \frac{\partial V}{\partial \alpha}\dot{\theta}\right)\sin\theta + V\cos(\theta)\dot{\theta}\right]$$

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\cos\theta\right) = \frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\right)\cos\theta - \frac{\partial V}{\partial \alpha}(\sin\theta)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\right)\left(U(x), \alpha(\theta)\right) = \frac{\partial^2 V}{\partial \alpha \partial U}\frac{dU}{dx}\frac{dx}{dt} + \frac{\partial^2 V}{\partial \alpha^2}\frac{d\alpha}{d\theta}\frac{d\theta}{dt}$$

($\frac{dx}{dt} \to V\cos\theta$, $\frac{d\alpha}{d\theta} \to 1$)

$$= \frac{\partial^2 V}{\partial \alpha \partial U}\frac{dU}{dx}V\cos\theta + \frac{\partial^2 V}{\partial \alpha^2}\dot{\theta}$$

So,

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \alpha}\cos\theta - V\sin\theta\right) = \left[\frac{\partial^2 V}{\partial\alpha\partial u}\frac{du}{dx}V\cos\theta + \frac{\partial^2 V}{\partial\alpha^2}\dot{\theta}\right]\cos\theta - \frac{\partial V}{\partial\alpha}(\sin\theta)\dot{\theta}$$
$$- \left[\frac{\partial V}{\partial u}\frac{du}{dx}V\cos\theta + \frac{\partial V}{\partial\alpha}\dot{\theta}\right]\sin\theta - V(\cos\theta)\dot{\theta}$$

$$= \dot{\theta}\left[\frac{\partial^2 V}{\partial\alpha^2}\cos\theta - 2\frac{\partial V}{\partial\alpha}\sin\theta - V\cos\theta\right] + \frac{\partial^2 V}{\partial\alpha\partial u}\frac{du}{dx}V\cos^2\theta$$
$$- \frac{\partial V}{\partial u}\frac{du}{dx}V\cos\theta\sin\theta$$

$$\therefore V\left[\frac{\partial^2 V}{\partial\alpha^2}\cos\theta - 2\frac{\partial V}{\partial\alpha}\sin\theta - V\cos\theta\right]\dot{\theta} + \frac{\partial^2 V}{\partial\alpha\partial u}\frac{du}{dx}V^2\cos^2\theta$$
$$- 2\left(\frac{\partial V}{\partial\alpha}\cos\theta - V\sin\theta\right)\left(\frac{\partial V}{\partial u}\frac{du}{dx}V\cos\theta + \frac{\partial V}{\partial\alpha}\dot{\theta}\right) - \frac{\partial V}{\partial u}\frac{du}{dx}V^2\sin\theta\cos\theta$$

Expand $2\left(\frac{\partial V}{\partial\alpha}\cos\theta - V\sin\theta\right)\left(\frac{\partial V}{\partial u}\frac{du}{dx}V\cos\theta + \frac{\partial V}{\partial\alpha}\dot{\theta}\right)$

$$= 2\frac{\partial V}{\partial\alpha}\frac{\partial V}{\partial u}\frac{du}{dx}V\cos^2\theta + 2\left(\frac{\partial V}{\partial\alpha}\right)^2(\cos\theta)\dot{\theta} - 2\frac{\partial V}{\partial u}\frac{du}{dx}\sin\theta\cos\theta$$
$$- 2\frac{\partial V}{\partial\alpha}V\sin\theta\,\dot{\theta}$$

$$= 2\frac{\partial V}{\partial\alpha}\frac{\partial V}{\partial u}\frac{du}{dx}V\cos^2\theta + 2\frac{\partial V}{\partial\alpha}\left(\frac{\partial V}{\partial\alpha}\cos\theta - V\sin\theta\right)\dot{\theta} - 2\frac{\partial V}{\partial u}\frac{du}{dx}V^2\sin\theta\cos\theta$$

$$\therefore \left\{V\left[\frac{\partial^2 V}{\partial\alpha^2}\cos\theta - 2\frac{\partial V}{\partial\alpha}\sin\theta - V\cos\theta\right] - 2\frac{\partial V}{\partial\alpha}\left(\frac{\partial V}{\partial\alpha}\cos\theta - V\sin\theta\right)\right\}\dot{\theta}$$
$$+ \frac{\partial^2 V}{\partial\alpha\partial u}\frac{du}{dx}V^2\cos^2\theta - \frac{\partial V}{\partial u}\frac{\partial u}{\partial x}V^2\sin\theta\cos\theta + 2\frac{\partial V}{\partial u}\frac{du}{dx}V^2\sin\theta\cos\theta$$
$$- 2\frac{\partial V}{\partial\alpha}\frac{\partial V}{\partial u}\frac{du}{dx}V\cos^2\theta = 0$$

$$\left\{ V\left[\frac{\partial^2 V}{\partial \alpha^2}\cos\theta - 2\frac{\partial V}{\partial \alpha}\sin\theta - V\cos\theta\right] - 2\frac{\partial V}{\partial \alpha}\left(\frac{\partial V}{\partial \alpha}\cos\theta - V\sin\theta\right)\right\}\dot{\theta} =$$

$$-\frac{\partial V}{\partial u}\frac{du}{dx}V^2\sin\theta\cos\theta - \frac{\partial^2 V}{\partial\alpha\partial u}\frac{du}{dx}V^2\cos^2\theta + 2\frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial u}\frac{du}{dx}V\cos^2\theta$$

$$= \left[\frac{\partial^2 V}{\partial \alpha^2}V\cos\theta - V^2\cos\theta - 2\left(\frac{\partial V}{\partial \alpha}\right)^2\cos\theta\right]\dot{\theta}$$

$$\left[\frac{\partial^2 V}{\partial \alpha^2}V\cos\theta - V^2 - 2\left(\frac{\partial V}{\partial \alpha}\right)^2\right]\dot{\theta} =$$

$$-\frac{\partial V}{\partial u}\frac{du}{dx}V^2\sin\theta - \frac{\partial^2 V}{\partial\alpha\partial u}\frac{du}{dx}V^2\cos\theta + 2\frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial u}\frac{du}{dx}V\cos\theta$$

$$\therefore \left[\frac{\partial^2 V}{\partial \alpha^2}V - V^2 - 2\left(\frac{\partial V}{\partial \alpha}\right)^2\right]\dot{\theta} = 2\frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial u}\frac{du}{dx}V\cos\theta - \frac{\partial V}{\partial u}\frac{du}{dx}V^2\sin\theta - \frac{\partial^2 V}{\partial\alpha\partial u}\frac{du}{dx}V^2\cos\theta$$

If V is Isotropic

$$\dot{\theta} = \frac{du}{dx}\frac{dV}{du}\sin\theta, \quad \theta(0) \text{ iterate}$$

Appendix B: *Matlab Code*

Isotropic Polar

M-File

```
function zdot = zdot(t,z)
x = z(1); y = z(2); theta = z(3); eps = 0.05; wbar = 5.0; L = 1.0;
v = -wbar * (1.0- (1.0 - eps) * x / L); vel = -v;
% wbar and L are given in knots
zdot = [vel * cos(theta); vel * sin(theta); -((1 - eps) * wbar / L) * sin(theta)];
```

Command

```
>>[t,z] = ode23(@zdot, [0.0:0.001:0.83]', [0.0 0.0 -1.452*pi/3]')
```

Non-Isotropic (Realistic) Polar

M-File

```
function zdotreal = zdotreal(t,z)
x = z(1); y = z(2); theta = z(3); eps = 0.05; wbar = 5.0; L = 1.0;
Uref = 16.65; %knots
c0=4.8401; c1=3.0336; c2=-4.7994; c3=0.9402; c4=1.0479; c5=-0.3765;
U = -wbar * (1.0- (1.0 - eps) * x / L);
alpha = theta + pi/2;
Ux = -((1 - eps) * wbar / L);
%Ux is the derivative of U WRT x
V = (U/Uref) * (c0 + c1*alpha + c2*alpha^2 + c3*alpha^3 + c4*alpha^4 + c5*alpha^5);
Va= (U/Uref) * (c1 + 2*c2*alpha + 3*c3*alpha^2 + 4*c4*alpha^3 + 5*c5*alpha^4);
%Va is the partial of V WRT alpha
Vaa= (U/Uref) * (2*c2 + 6*c3*alpha + 12*c4*alpha^2 + 20*c5*alpha^3);
%Vaa is the 2nd partial of V WRT alpha
VU= (c0 + c1*alpha + c2*alpha^2 + c3*alpha^3 + c4*alpha^4 + c5*alpha^5)/Uref;
%Vu is the partial of V WRT U
VaU = (c1 + 2*c2*alpha + 3*c3*alpha^2 + 4*c4*alpha^3 + 5*c5*alpha^4)/Uref;
%Vau is the partial of V WRT a and U
vel = -V;
numerator = -VU * Ux * V^2 * sin(theta) - VaU * Ux * V^2 * cos(theta) + 2 * Va * VU * Ux * V *
cos(theta);
denominator = Vaa * V - V^2 - 2 * (Va)^2;
zdotreal = [vel * cos(theta); vel * sin(theta); numerator/denominator];
```

Command

```
>> [t,z] = ode23(@zdotreal, [0.0:0.001:2.7680]', [0.0 0.0 -1.12415*pi/3]')
```