

## CSUMS Seminar 9: Finite Difference Methods for PDEs

The backward difference in time central difference in space discretization for the diffusion equation

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}$$

has the form

$$\frac{u_i^n - u_i^{n-1}}{k} = a \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.$$

The above equation can be rewritten as

$$(1 + 2\beta)u_i^n - \beta u_{i+1}^n - \beta u_{i-1}^n = u_i^{n-1}, \quad (1)$$

where  $\beta = \frac{ak}{h^2}$ .

1. Use a von Neumann analysis on (1) to determine a relationship between  $k$  and  $h$  for which the numerical scheme will be stable.
2. Assume that the Dirichlet boundary conditions for the diffusion equation are

$$u(0, t) = \alpha u(L, 2) = \delta$$

for constants  $\alpha$  and  $\delta$ . As discussed in class, the discretization (1) must be written in matrix form and solved as a linear system. Write out the matrix form  $\mathbf{A}\mathbf{u} = \mathbf{b}$  that incorporates the boundary conditions above. In other words, write out the form of  $\mathbf{A}$  and  $\mathbf{b}$  for a given time step  $n$  and for an arbitrary number of  $M + 2$  spatial points (including boundary conditions). Note that the process for this problem is similar to what was done in class for finite difference methods for ordinary differential equations.